

HOFER-ZEHNDER CAPACITY AND BRUHAT GRAPH

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ABSTRACT. We find bounds for the Hofer-Zehnder capacity of coadjoint orbits of compact Lie groups with respect to the Kostant–Kirillov–Souriau symplectic form in terms of the combinatorics of their Bruhat graph. We show that our bounds are sharp for coadjoint orbits of the unitary group and equal to the diameter of a weighted Cayley graph.

1. INTRODUCTION

The Gromov non-squeezing theorem in symplectic geometry states that is not possible to embed symplectically a ball into a cylinder of smaller radius, although this can be done with volume preserving embeddings [11]. Hence, the biggest radius of a ball that can be symplectically embedded into a symplectic manifold can be used as a way to measure the “symplectic size” of the manifold. We call this radius *Gromov’s width*.

The Gromov width as a symplectic invariant is extended through the notion of *symplectic capacity* whose axiomatic formulation is due to Ekeland and Hofer [4], [5]. An important example of capacity is the *Hofer-Zehnder capacity* [15]. The Hofer-Zehnder capacity of a closed symplectic manifold (M, ω) is defined as

$$c_{\text{HZ}}(M, \omega) := \sup \left\{ \max H - \min H \mid H : M \rightarrow \mathbb{R} \text{ slow} \right\},$$

where a Hamiltonian $H : M \rightarrow \mathbb{R}$ is slow if the periodic trajectories of its Hamiltonian flow are either constant or have period greater or equal to one. In comparison with the Gromov width, the Hofer-Zehnder capacity measures the size of a symplectic manifold in a Hamiltonian dynamic way

In this paper, we are interested in computing bounds for the Hofer-Zehnder capacity of coadjoint orbits of compact Lie groups with respect to their Kostant–Kirillov–Souriau form. We summarize the main results in this paper in the following theorem.

Theorem 1.1. *Let G be a compact connected simple Lie group with Lie algebra \mathfrak{g} . We identify the Lie algebra \mathfrak{g} with its dual \mathfrak{g}^* via an adjoint invariant inner product. Let $T \subset G$ be a maximal torus. For $\lambda \in \mathfrak{t} \subset \mathfrak{g}$, let \mathcal{O}_λ be the coadjoint orbit passing through λ and ω_λ be the Kostant–Kirillov–Souriau form defined*

on \mathcal{O}_λ . Let R be the corresponding system of roots and S be a choice of simple roots.

For a positive root β we write

$$\beta = \sum_{\alpha \in S} n_{\beta\alpha} \alpha$$

for some nonnegative integer $n_{\beta\alpha}$. We denote by $\check{\beta}$ the coroot associated with a root β .

Let $W = N_G(T)/T$ be the Weyl group relative to T and w_0 be the longest element in W relative to the set of simple roots S . If there exist positive roots $\alpha_1, \dots, \alpha_r$ such that

$$w_0 = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_r},$$

then we obtain the following bounds for the Hofer-Zehnder capacity of $(\mathcal{O}_\lambda, \omega_\lambda)$

$$(1) \quad \max_{\alpha \in S} \left\{ \sum_{k=1}^r \frac{n_{\alpha_k \alpha}}{n_{\rho \alpha}} |\langle \lambda, \check{\alpha}_k \rangle| \right\} \leq c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \sum_{k=1}^r |\langle \lambda, \check{\alpha}_k \rangle|,$$

here ρ denotes the highest positive root.

In the proof of the nonsqueezing theorem, Gromov noted that the Gromov width of a symplectic manifold is constrained by the existence of pseudoholomorphic curves [11]. The relation between pseudoholomorphic curves and the Hofer-Zehnder capacity was also observed by several authors in the context of the Weinstein conjecture (see e.g Floer, Hofer and Viterbo [7], Hofer and Viterbo [14], Liu and Tian [19]). This relation appears more explicit in a result of G. Lu that bounds the Hofer-Zehnder capacity of a symplectic manifold when it has a nonzero Gromov-Witten invariant with two point constraints [21], [22]. In this paper, we use G. Lu's result to bound from above the Hofer-Zehnder capacity of coadjoint orbits of compact Lie groups.

The *Bruhat graph* (also known as *moment graph* or *GKM graph*) of a coadjoint orbit is the graph whose vertices and edges are in one to one correspondence with the points and irreducible invariant curves that are invariant with respect to the action of a maximal torus on the coadjoint orbit. A result of Fulton and Woodward states that the minimal degrees appearing in the nonvanishing Gromov-Witten invariants of a coadjoint orbit can be interpreted in terms of paths of its Bruhat graph [9]. The main goal of the present paper is to point out the relation between the Bruhat graph and the Hofer-Zehnder capacity of coadjoint orbits of compact Lie groups.

If we weight the edges of the Bruhat graph of the coadjoint orbit with the symplectic area of the curves that they represent, then the right hand side of the inequality (1) appearing in the Main theorem can be reinterpreted as the following inequality

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \text{diameter weighted Bruhat graph of } (\mathcal{O}_\lambda, \omega_\lambda)$$

In this paper we show that the previous inequality is sharp for coadjoint orbits of the unitary group.

Theorem 1.2. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and assume that $\lambda_1 \geq \dots \geq \lambda_n$. Let*

$$\mathcal{H}_\lambda := \{A \in M_n(\mathbb{C}) : A^* = -A, \text{ spectrum } A = i\lambda\}.$$

We identify \mathcal{H}_λ with a coadjoint orbit of $U(n)$ and endow it with a symplectic form ω_λ coming from the Kostant-Kirillov-Souriau form.

Let us consider the weighted Cayley graph of the symmetric group S_n where two permutations are joined by an edge of weight $|\lambda_i - \lambda_j|$ if they differ by a trasposition (i, j) . Then

$$\begin{aligned} c_{\text{HZ}}(\mathcal{H}_\lambda, \omega_\lambda) &= \frac{1}{2} \sum_{k=1}^n |\lambda_k - \lambda_{n-k+1}| \\ &= \text{diameter of the weighted Cayley graph of } S_n. \end{aligned}$$

The Hofer-Zehnder capacity of a coadjoint orbit of the unitary group is in contrast with its Gromov width that is equal to the smallest weight of the weighted Cayley graph of S_n defined in the previous theorem (see e.g. Caviedes [3], Pabiniak [27]). In particular, the Hofer-Zehnder capacity of a coadjoint orbit isomorphic with a projective space coincides with its Gromov width and the Hofer-Zehnder capacity of a coadjoint orbit isomorphic with a Grassmannian manifold is equal to an integer multiple of its Gromov's width, and we recover results of Hofer and Viterbo for the projective space [14] and G. Lu for the Grassmannian manifold [21].

We suggest that the reader compares our results with the ones of Loi, Mossa and Zuddas [20] where they estimate the Hofer-Zehnder capacity of Hermitian symmetric spaces and with the ones of Hwang and Suh [17] where they compute the Hofer-Zehnder capacity of symplectic manifolds with Hamiltonian semifree circle actions in terms of their moment map.

This paper is organized as follows: in the second section, we review the definition of Hofer-Zehnder capacity of a symplectic manifold and state G.Lu's theorem that bounds the Hofer-Zehnder capacity of a symplectic manifold in terms of its Gromov-Witten invariants. In the third section, we recall background on the geometry of coadjoint orbits of compact Lie groups. In the fourth section, we define the Bruhat graph and indicate its relation with the Hofer-Zehnder capacity of coadjoint orbits. In the fifth section, we compute the Hofer-Zehnder capacity of coadjoint orbits of the unitary group. In the sixth section, we recall results of Postnikov concerning the minimal degrees of paths in the Bruhat graph, and explain how they can be used to find more optimal upper bounds for the Hofer-Zehnder capacity of regular coadjoint orbits. In the seventh section, we explain how to bound from below the Hofer-Zehnder capacity of a coadjoint orbit using the moment map of the Hamiltonian group action of a maximal torus.

In the eight section we write explicitly our bounds for every simple compact Lie group according to the type.

2. HOFER-ZEHNDER CAPACITY AND GROMOV-WITTEN INVARIANTS

Let (M, ω) be a closed symplectic manifold. A *Hamiltonian* is a smooth function $H : (\mathbb{R}/\mathbb{Z}) \times M \rightarrow \mathbb{R}$. A Hamiltonian is *autonomous* if it is time-independent. The Hamiltonian vector field of H is the time-dependent vector field X_H defined by

$$d(H(t, \cdot)) = \iota_{X_H(t, \cdot)} \omega.$$

The *oscillation* of an autonomous Hamiltonian $H : M \rightarrow \mathbb{R}$ is

$$\text{osc } H := \max H - \min H$$

A Hamiltonian function $H : M \rightarrow \mathbb{R}$ is *slow* if all periodic orbits of the Hamiltonian vector field X_H of period less than one are constant.

The *Hofer-Zehnder capacity* of (M, ω) is defined as

$$c_{\text{HZ}}(M, \omega) := \sup \{ \text{osc } H \mid H : M \rightarrow \mathbb{R} \text{ slow} \}$$

Let J be a Fredholm regular almost complex structure compatible with ω , (see the definition for instance in McDuff and Salamon [23]). Let d be a class in $H_2(M; \mathbb{Z})$. We denote by $\text{GW}_{d,k}(a_1, a_2, \dots, a_k)$ the *Gromov-Witten invariant* that roughly speaking counts the number of J -holomorphic spheres in M in the class $d \in H_2(M; \mathbb{Z})$ that meet cycles representing the homology classes $a_1, a_2, \dots, a_k \in H_*(M; \mathbb{Z})$. The following theorem due to G. Lu bounds from above the Hofer-Zehnder capacity of a closed symplectic manifold in terms of its Gromov-Witten invariants [21].

Theorem 2.1 (G. Lu [21]). *Let (M, ω) be a closed symplectic manifold. Suppose that (M, ω) admits a nonzero Gromov-Witten invariant of the form*

$$\text{GW}_{d,k}([\text{pt}], [\text{pt}], a_2, \dots, a_k)$$

for some $k \in \mathbb{Z}_{\geq 1}$, $d \in H_2(M; \mathbb{Z})$ and $a_2, \dots, a_k \in H_(M; \mathbb{Z})$. Then*

$$c_{\text{HZ}}(M, \omega) \leq \omega(d)$$

3. GEOMETRY OF COADJOINT ORBITS

In this section we establish the Lie theoretical convention that is used through the rest of the paper. Most of the material can be found in the classical literature that is concerned about the geometry and topology of coadjoint orbits such as Bernstein, Gelfand and Gelfand [1] and Kirillov [18].

Let G be a compact Lie group, \mathfrak{g} be its Lie algebra and \mathfrak{g}^* be the dual of \mathfrak{g} . Let (\cdot, \cdot) denote an adjoint invariant inner product defined on \mathfrak{g} . We identify the Lie algebra \mathfrak{g} and its dual \mathfrak{g}^* via this inner product. Let $\lambda \in \mathfrak{g}^*$ and $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ be

the coadjoint orbit passing through λ . Let ω_λ be the *Kostant-Kirillov-Souriau form* defined on \mathcal{O}_λ by

$$\omega_\lambda(\hat{X}, \hat{Y}) = \langle \lambda, [X, Y] \rangle \quad X, Y \in \mathfrak{g},$$

where \hat{X}, \hat{Y} are the vector fields on \mathfrak{g}^* generated by the coadjoint action of G . The form ω_λ is closed and non-degenerate thus defining a symplectic structure on \mathcal{O}_λ .

We denote by $G_{\mathbb{C}}$ the complexification of the Lie group G . Let $P \subset G_{\mathbb{C}}$ be a parabolic subgroup of $G_{\mathbb{C}}$ such that $\mathcal{O}_\lambda \cong G_{\mathbb{C}}/P$. The quotient of complex Lie groups $G_{\mathbb{C}}/P$ allows us to endow \mathcal{O}_λ with a complex structure J compatible with ω_λ so the triple $(\mathcal{O}_\lambda, \omega_\lambda, J)$ is a Kähler manifold. The almost complex structure J is Fredholm regular (see e.g. McDuff and Salamon [23, Proposition 7.4.3]).

Let $T \subset G$ be a maximal torus and \mathfrak{t} denote its Lie algebra. Let $R \subset \mathfrak{t}^*$ be the root system of T in G so

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha},$$

where

$$\mathfrak{g}_{\alpha} := \{x \in \mathfrak{g}_{\mathbb{C}} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{t}_{\mathbb{C}}\}$$

is the root space associated with the root $\alpha \in R$. Let $R^+ \subset R$ be a choice of positive roots with simple roots $S \subset R^+$. Let $B \subset G_{\mathbb{C}}$ be the Borel subgroup with Lie algebra

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\alpha}$$

Each root $\alpha \in R$ has a coroot $\check{\alpha} \in \mathfrak{t}$. The coroot $\check{\alpha}$ is identified with $\frac{2\alpha}{\langle \alpha, \alpha \rangle} \in \mathfrak{t}$ via the invariant inner product (\cdot, \cdot) . The system of coroots is the set $\check{R} = \{\check{\alpha} : \alpha \in R\}$ and the simple coroots is the set $\check{S} = \{\check{\alpha} : \alpha \in S\}$.

Every root $\alpha \in R$ defines a reflection s_{α} on \mathfrak{t}^* given by

$$\begin{aligned} s_{\alpha} : \mathfrak{t}^* &\rightarrow \mathfrak{t}^* \\ t &\mapsto t - \langle t, \check{\alpha} \rangle \alpha, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard pairing $\langle \cdot, \cdot \rangle : \mathfrak{t}^* \otimes \mathfrak{t} \rightarrow \mathbb{R}$. The group W generated by the set of reflections $\{s_{\alpha}\}_{\alpha \in R}$ is the *Weyl group* of G relative to T . It is known that the Weyl group W can be identified with $N_G(T)/T$.

We can associate to the parabolic subgroup $P \subset G_{\mathbb{C}}$ a subset of simple roots defined by

$$S_P := \{\alpha \in S : \langle \lambda, \check{\alpha} \rangle \neq 0\}.$$

We denote by W_P the Weyl group of P generated by the set of simple roots S_P . The Weyl group W_P is identified with $N_P(T)/T$. Also, set $R_P := R \cap \mathbb{Z}S_P$ and $R_P^+ := R^+ \cap \mathbb{Z}S_P$, where $\mathbb{Z}S_P = \text{span}_{\mathbb{Z}}(S_P)$.

The *Weyl chamber* relative to the set of simple roots S is the convex polyhedron

$$\mathfrak{t}_+^* := \{\gamma \in \mathfrak{t}^* : \langle \gamma, \check{\alpha} \rangle \geq 0 \text{ for all } \alpha \in S\}$$

The vector space \mathfrak{t}^* can be decomposed as the union of convex polyhedrons

$$\mathfrak{t}^* = \bigcup_{w \in W} w(\mathfrak{t}_+^*)$$

whose interiors are disjoint. For the coadjoint orbit \mathcal{O}_λ , there exists $\lambda' \in \mathfrak{t}_+^*$ such that $\mathcal{O}_\lambda \cap \mathfrak{t}^* = \{w(\lambda')\}_{w \in W}$. Indeed, the group T acts hamiltonially on \mathcal{O}_λ with moment map $\phi : \mathcal{O}_\lambda \hookrightarrow \mathfrak{t}^*$ equals to the composition of the projection map $\mathfrak{g}^* \rightarrow \mathfrak{t}^*$ with the inclusion map $\mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$. The image of ϕ is the convex hull of $\{w(\lambda)\}_{w \in W}$. We always assume in what follows that $\lambda \in \mathfrak{t}_+^*$.

For $w \in W$, the *length* $l(w)$ of w is defined as the minimum number of simple reflections $s_\alpha \in W$, $\alpha \in S$, whose product is w . The Weyl group W has a unique longest element that we denote by w_0 .

Let $B^{\text{op}} := w_0 B w_0 \subset G_{\mathbb{C}}$ be the *Borel subgroup opposite to B* . For $w \in W/W_P$, let $X(w) := \overline{BwP/P} \subset G_{\mathbb{C}}/P$ and $Y(w) := \overline{B^{\text{op}}wP/P} \subset G_{\mathbb{C}}/P$ be the *Schubert variety* and the *opposite Schubert variety* associated with w , respectively. We denote by σ_w and $\check{\sigma}_w$ the fundamental classes in the homology group $H_*(G_{\mathbb{C}}/P; \mathbb{Z})$ of $Y(w)$ and $X(w)$, respectively. Note that $\check{\sigma}_w = \sigma_{w_0 w} = \sigma_{\check{w}}$, where $\check{w} := w_0 w$. The set of Schubert classes $\{\sigma_w\}_{w \in W/W_P}$ forms a free \mathbb{Z} -basis of $H_*(G_{\mathbb{C}}/P; \mathbb{Z})$, and the set of Schubert classes $\{\check{\sigma}_w\}_{w \in W/W_P}$ is the dual basis of $\{\sigma_w\}_{w \in W/W_P}$ with respect to the Poincaré intersection pairing.

The *Bruhat order* \prec on W/W_P is defined by $u \prec v$ if $X(u) \subset X(v)$.

4. BRUHAT GRAPH

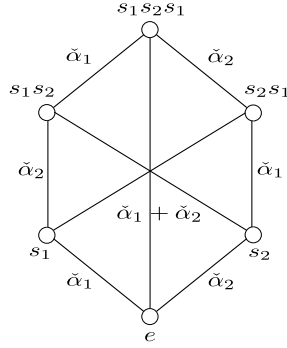
In this section we define the Bruhat graph and indicate its relation with the Hofer-Zehnder capacity of a coadjoint orbit of compact Lie group.

We keep the convention of the last section. Let G be a compact Lie group. Let $T \subset G$ be a maximal torus, $B \subset G_{\mathbb{C}}$ be a Borel subgroup and $P \subset G_{\mathbb{C}}$ be a parabolic subgroup such that $T \subset B \subset P$. Let R and S be the system of roots and simple roots determined by T and B , respectively.

The *Bruhat graph* is the graph on W/W_P where two elements are joined by an edge if they differ by one reflection. More precisely, there is an edge joining u with v if and only if there exists a positive root $\alpha \in R^+ - R_P^+$ such that

$$v = u \cdot s_\alpha \mod W_P.$$

In Figure 1, we show the Bruhat graph of the Weyl group of $U(3)$. The standard set of simple roots of $U(3)$ is $\{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3\} \subset \mathbb{R}^3$, where $\{e_1, e_2, e_3\}$ denotes the standard basis of \mathbb{R}^3 . The Weyl group of $U(3)$ is the symmetric group S_3 generated by the simple reflections $s_1 := s_{e_1 - e_2} = (12)$, $s_2 := s_{e_2 - e_3} = (23)$.

FIGURE 1. Bruhat graph of S_3

The vertices and edges of the Bruhat graph are in one-to-one correspondence with the T -fixed points and irreducible T -invariant curves of $G_{\mathbb{C}}/P$, respectively. The collection of cosets $\{wP\}_{w \in W/W_P}$ is the set of all T -fixed points of $G_{\mathbb{C}}/P$. For each positive root $\alpha \in R^+ - R_P^+$ there is a unique irreducible T -invariant curve C_α that contains $1 \cdot P$ and $s_\alpha \cdot P$. Indeed, $C_\alpha := \text{Sl}(2, \mathbb{C})_\alpha \cdot P/P$ where $\text{Sl}(2, \mathbb{C})_\alpha \subset G_{\mathbb{C}}$ is the subgroup of $G_{\mathbb{C}}$ with Lie algebra

$$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}_{\mathbb{C}}^*.$$

We identify the homology group $H_2(G_{\mathbb{C}}/P; \mathbb{Z})$ with $\mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$ via the transformation

$$[C_\alpha] \mapsto \check{\alpha} + \mathbb{Z}\check{S}_P$$

(see e.g. Fulton and Woodward [9]). We weight the edges of the Bruhat graph with elements in $H_2(G_{\mathbb{C}}/P; \mathbb{Z}) \cong \mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$. If u and v differ by the reflection s_α , then the weight of the corresponding edge is $\check{\alpha} + \mathbb{Z}\check{S}_P$. We define an ordering on the set of degrees $H_2(G_{\mathbb{C}}/P; \mathbb{Z})$ as follows: for $c, d \in H_2(G_{\mathbb{C}}/P; \mathbb{Z})$, we say that $c \leq d$ if there exists $n_\alpha \in \mathbb{Z}_{\geq 0}$ such that

$$d - c = \sum_{\alpha \in S - S_P} n_\alpha \check{\alpha} \mod \mathbb{Z}\check{S}_P$$

A *chain* from u to v in W/W_P is a sequence $u_0, u_1, \dots, u_r \in W/W_P$ such that u_i and u_{i-1} are adjacent for $1 \leq i \leq r$, $u \prec u_0$ and $u_r \prec \check{v} = w_0 v$. A chain from u to v corresponds to a sequence of T -invariant curves C_1, C_2, \dots, C_r with C_1 meeting $Y(u)$ and C_r meeting $X(\check{v})$. The *degree* of the chain is the sum of the classes $[C_i]$ of the curves. A *path* from u to v in W/W_P is a sequence $u_0, u_1, \dots, u_r \in W/W_P$ such that u_i and u_{i-1} are adjacent for $1 \leq i \leq r$, $u = u_0$ and $u_r = v$. A path in the Bruhat graph coincides with the standard notion of path in graph theory. The degree of a path is defined in the same way as the degree of a chain.

The following result due to Fulton and Woodward establishes the relation between chains in the Bruhat graph of W/W_P and the Gromov-Witten invariants of $G_{\mathbb{C}}/P$.

Theorem 4.1 (Fulton-Woodward [9]). *Let $u, v \in W/W_P$ and $d \in H_2(G_{\mathbb{C}}/P; \mathbb{Z})$. The following are equivalent:*

- (1) *There is a chain of degree $c \leq d$ between u and v in the Bruhat graph of W/W_P .*
- (2) *There exists a sequence C_0, C_1, \dots, C_r of T -invariant curves with C_0 meeting $Y(u)$ and C_r meeting $X(v)$, with C_{i-1} meeting C_i for $1 \leq i \leq r$, and with $\sum_{i=0}^r [C_i] \leq d$.*
- (3) *There exist a degree $c \leq d$ and w in W/W_P such that*

$$\text{GW}_{c,3}(\sigma_u, \sigma_v, \sigma_w) \neq 0$$

Now we state the relation between chains in the Bruhat graph and the Hofer-Zehnder capacity of coadjoint orbits.

Theorem 4.2. *Assume that λ lies in the Weyl chamber \mathfrak{t}_+^* relative to the set of simple roots S and the coadjoint orbit \mathcal{O}_λ passing through λ is isomorphic with $G_{\mathbb{C}}/P$. We endow the coadjoint orbit \mathcal{O}_λ with its Kostant-Kirillov-Souriau symplectic form ω_λ . Then*

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \min_d \omega_\lambda(d),$$

where the minimum is taken over all the degrees $d \in H_2(\mathcal{O}_\lambda; \mathbb{Z}) \cong \mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$ of paths joining $[e]$ with $[w_0]$ in the Bruhat graph of W/W_P .

Proof. Let d be minimal among the set of all degrees of paths joining P/P with w_0P/P in the Bruhat graph. According to the Theorem 4.1 of Fulton and Woodward, there exists $u \in W/W_P$ such that

$$\text{GW}_{d,3}([\text{pt}], [\text{pt}], \sigma_u) \neq 0$$

By Theorem 2.1 of G.Lu,

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \omega_\lambda(d),$$

and we are done. \square

Remark 4.3. A path in the Bruhat graph joining P/P with w_0P/P is the same as an ordered sequence of positive roots $\alpha_1, \dots, \alpha_r \in R - R_P$ such that

$$w_0 = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_r} \mod W_P$$

In this case, the path in the Bruhat graph is given by the sequence

$$e \xrightarrow{\alpha_1} s_{\alpha_1} \xrightarrow{\alpha_2} s_{\alpha_1} s_{\alpha_2} \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{r-1}} s_{\alpha_1} \cdot \dots \cdot s_{\alpha_{r-1}} \xrightarrow{\alpha_r} w_0$$

The degree of the corresponding sequence of T -invariant curves is equal to

$$\sum_{i=1}^r \check{\alpha}_i$$

The symplectic area of the curve C_α with respect to the Kostant-Kirillov-Souriau form is equal to $\langle \lambda, \check{\alpha} \rangle$ (see e.g. McDuff and Tolman [24][Lemma 3.9]), hence the symplectic area of the above sequence of T -invariant curves is equal to

$$\sum_{i=1}^r \langle \lambda, \check{\alpha}_i \rangle.$$

5. HOFER-ZEHNDER CAPACITY OF COADJOINT ORBITS OF $U(n)$

In this section we compute the Hofer-Zehnder capacity of coadjoint orbits of the unitary group.

We denote by $U(n)$ the set of $n \times n$ unitary matrices and by $\mathfrak{u}(n)$ its Lie algebra. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ and

$$\mathcal{H}_\lambda := \{A \in \mathfrak{u}(n) : A^* = -A, \text{ spectrum } A = i\lambda\}$$

The unitary group $U(n)$ acts on \mathcal{H}_λ by conjugation. We identify the set of skew-Hermitian matrices \mathcal{H}_λ with a regular coadjoint orbit of $U(n)$ via the pairing

$$\begin{aligned} \mathfrak{u}(n) \times \mathfrak{u}(n) &\rightarrow \mathbb{R} \\ (X, Y) &\mapsto \text{Trace}(XY) \end{aligned}$$

We denote by ω_λ the symplectic form obtained by identifying \mathcal{H}_λ with a coadjoint orbit of $U(n)$.

Let $T = U(1)^n \subset U(n)$ be the maximal torus of diagonal matrices in $U(n)$. We identify the Lie algebra of T with \mathbb{R}^n and we denote by $\{e_1, \dots, e_n\}$ the standard basis of \mathbb{R}^n . The system of positive roots associated with the torus T is the set of vectors

$$\{\alpha_{i,j} := e_i - e_j\}_{1 \leq i < j \leq n} \subset \mathfrak{t} \cong \mathfrak{t}^*$$

The standard system of simple roots is the set

$$\{\alpha_i := \alpha_{i,i+1}\}_{1 \leq i < n}$$

The corresponding Dynkin diagram is shown in Figure 2

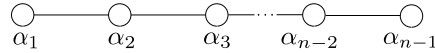


FIGURE 2. Dynkin diagram of A_n

Any T -fixed point of \mathcal{H}_λ is a permutation of the diagonal matrix $i(\lambda_1, \dots, \lambda_n)$. Two T -fixed points of \mathcal{H}_λ are joined by one irreducible T -invariant curve if they differ by one transposition.

Theorem 5.1. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ and assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then,*

$$c_{\text{HZ}}(\mathcal{H}_\lambda, \omega_\lambda) = \frac{1}{2} \sum_{k=1}^n |\lambda_k - \lambda_{n-k+1}|$$

Proof. According to Theorem 4.1, in order to find an upper bound for the Hofer Zehnder capacity of \mathcal{H}_λ , we want to find a path of irreducible T -invariant curves joining the diagonal matrix $i(\lambda_1, \lambda_2, \dots, \lambda_n)$ with the diagonal matrix $i(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$. Let us consider the path given by the following sequence

$$\begin{aligned} i(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) &\xrightarrow{(1, n)} i(\lambda_n, \lambda_2, \dots, \lambda_{n-1}, \lambda_1) \\ &\xrightarrow{(2, n-1)} \dots \rightarrow i(\lambda_n, \lambda_{n-1}, \dots, \lambda_2, \lambda_1), \end{aligned}$$

The degree of this path is equal to

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \check{\alpha}_{k, n-k+1}$$

and its symplectic area is equal to

$$\omega_\lambda \left(\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \check{\alpha}_{k, n-k+1} \right) = \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \langle \lambda, \check{\alpha}_{k, n-k+1} \rangle = \frac{1}{2} \sum_{k=1}^n |\lambda_k - \lambda_{n-k+1}|,$$

and thus

$$c_{\text{HZ}}(\mathcal{H}_\lambda, \omega_\lambda) \leq \frac{1}{2} \sum_{k=1}^n |\lambda_k - \lambda_{n-k+1}|.$$

Now we show that this inequality is sharp by constructing an admissible Hamiltonian function $H : \mathcal{H}_\lambda \rightarrow \mathbb{R}$ whose oscillation is equal to the right hand side of the inequality. The conjugation action of the torus T on \mathcal{H}_λ is Hamiltonian with moment map given by

$$\begin{aligned} \mu : \mathcal{H}_\lambda &\rightarrow i\mathbb{R}^n \\ A &\rightarrow \text{diagonal}(A) \end{aligned}$$

The image of the moment map μ is the convex hull of all possible permutations of the vector $i(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ (see e.g. Guillemin [12]).

For $t \in U(1)$ and $(m_1, \dots, m_n) \in \mathbb{Z}^n$, we use the convention

$$t^{(m_1, \dots, m_n)} := (t^{m_1}, \dots, t^{m_n}) \in T = U(1)^n \subset U(n).$$

Let

$$\beta = \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (e_k - e_{n-k+1})$$

and $S = \{t^\beta = (t, t, \dots, t^{-1}, t^{-1}) : t \in S^1\} \subset T$. The action of the circle S on \mathcal{H}_λ is Hamiltonian with moment map given by

$$\begin{aligned} \tilde{\mu} : \mathcal{H}_\lambda &\rightarrow i\mathbb{R} \\ A = (a_{ij}) &\mapsto \langle \mu(A), \beta \rangle = a_{1,1} - a_{n,n} + a_{2,2} - a_{n-1,n-1} + \dots \end{aligned}$$

The moment map image of $\tilde{\mu}$ is the interval

$$i \left[-\frac{1}{2} \sum_{k=1}^n |\lambda_k - \lambda_{n-k+1}|, \frac{1}{2} \sum_{k=1}^n |\lambda_k - \lambda_{n-k+1}| \right] \subset i\mathbb{R},$$

and thus the oscillation of $\text{Im}(\tilde{\mu})$ is equal to $\sum_{k=1}^n |\lambda_k - \lambda_{n-k+1}|$. Unfortunately, the function $\text{Im}(\tilde{\mu})$ is not slow. This is because, under the action of S on \mathcal{H}_λ , there are elements in \mathcal{H}_λ with non-trivial finite stabilizers. All possible stabilizer subgroups of S are either $\{1\}$, \mathbb{Z}_2 or S . If the stabilizer subgroup of a skew-Hermitian matrix in \mathcal{H}_λ is \mathbb{Z}_2 , the period of the orbit passing through the skew-Hermitian matrix is one half. Otherwise the skew-Hermitian matrix is either a S -fixed point or the period of the orbit passing through the skew-Hermitian matrix is one.

The Hamiltonian function $H = \frac{1}{2} \text{Im}(\tilde{\mu}) : \mathcal{H}_\lambda \rightarrow \mathbb{R}$ fixes this problem. The orbits of H are either constant or their periods are either one or two. So, the Hamiltonian H is slow, and

$$\text{osc}(H) = \frac{1}{2} \sum_{k=1}^n |\lambda_k - \lambda_{n-k+1}| \leq c_{\text{HZ}}(\mathcal{H}_\lambda, \omega_\lambda)$$

and we are done. \square

Remark 5.2. The Hofer-Zehnder capacity of the coadjoint orbit \mathcal{H}_λ is the same as the *diameter* of the weighted Cayley graph of S_n where two permutations are joined by an edge of weight $|\lambda_i - \lambda_j|$ if they differ by a trasposition (i, j) . In this weighted Cayley graph, the distance between the identity permutation e and any other permutation σ is given by the expression

$$d(1, \sigma) = \frac{1}{2} \sum_{i=1}^n |\lambda_i - \lambda_{\sigma(i)}|,$$

and we have the following rearrangement inequality

$$0 \leq \frac{1}{2} \sum_i |\lambda_i - \lambda_{\sigma(i)}| \leq \frac{1}{2} \sum_i |\lambda_i - \lambda_{n-i+1}|$$

(see e.g. Farnoud and Milenkovic [6][Theorem 22] for the first inequality, and Rinott [30] and Vince [34][Example 2] for the second).

As an illustrative example, Figure 3 shows the Bruhat graph associated with the set of skew-Hermitian matrices $\mathcal{H}_{(\lambda_1, \lambda_1, 0, 0)}$. We weight the edges of the graph with the symplectic areas of the corresponding T -invariant curves. All

the symplectic areas are equal to $|\lambda_1|$. The diameter of this weighted graph, that is the Hofer-Zehnder capacity of $\mathcal{H}_{(\lambda_1, \lambda_1, 0, 0)}$, is equal to $2|\lambda_1|$.

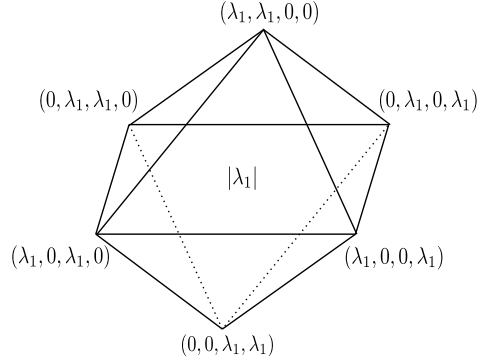


FIGURE 3. Bruhat graph of $\mathcal{H}_{(\lambda_1, \lambda_1, 0, 0)}$

6. UPPER BOUNDS FOR THE HOFER-ZEHNDER CAPACITY OF REGULAR COADJOINT ORBITS

According to the previous section, we can bound from above the Hofer-Zehnder capacity of a coadjoint orbit of a compact Lie group by considering paths in its Bruhat graph. In order to achieve an optimal upper bound for the Hofer-Zehnder capacity through this method, we want to determine the paths of minimal degree joining the identity element and the longest element of the Weyl group. A theorem due to Postnikov states that the minimal degree of such paths is unique for *regular* coadjoint orbits of compact Lie groups [29]. Recall that a coadjoint orbit of a compact Lie group is regular if the stabilizer subgroup of any element in the coadjoint orbit is a maximal torus.

In this section, we recall Postnikov's results and its combinatorial formulation in terms of the *quantum Bruhat graph* as it is done in [29]. We also give a criterion that allow us to characterize the path of minimal degree joining the identity element and the longest element in the Bruhat graph.

We follow the same convention as in Section 3. Let G be a compact Lie group. Let $T \subset G$ be a maximal torus, $B \subset G_{\mathbb{C}}$ be a Borel subgroup such that $T \subset B \subset P$. Let R and S be the system of roots and simple roots determined by T and B , respectively. Let W be the corresponding Weyl group and w_0 be the longest element in W relative to S . For a positive root α write

$$\check{\alpha} = \sum_{\beta \in S} \check{n}_{\alpha\beta} \check{\beta}$$

for some nonnegative integers $\check{n}_{\alpha\beta}$ and define the height of $\check{\alpha}$ as

$$\text{ht}(\check{\alpha}) := \sum_{\beta \in S} \check{n}_{\alpha\beta}$$

The *quantum Bruhat graph* of W is a directed graph on the elements of the Weyl group with weighted edges defined as follows: two elements $u, v \in W$ are connected by a directed edge $u \rightarrow v$ if and only if $v = us_\alpha$ and one of the following two conditions is satisfied

$$l(v) = l(u) + 1 \quad \text{or} \quad l(v) = l(u) + 1 - 2 \text{ht}(\check{\alpha})$$

If $l(v) = l(u) + 1$ then the degree of the edge equals 0, and if $l(v) = l(u) + 1 - 2 \text{ht}(\check{\alpha})$ then the degree of the edge equals $\check{\alpha}$. Note that two vertices can be connected with edges going in both directions. The *degree* of a directed path in the quantum Bruhat graph of W is the sum of degrees of its edges. The *length* of a directed path is the number of edges that it uses. A directed path in the quantum Bruhat graph from u to v is *shortest* if it has the minimal possible length among all direct paths from u to v . Figure 4 shows the quantum Bruhat graph of S_3 following the same convention as in Figure 1.

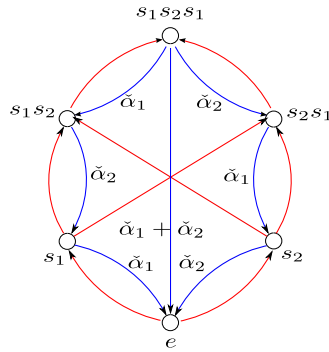


FIGURE 4. Quantum Bruhat graph of S_3

Now we recall the following result about the combinatorics of paths in the quantum Bruhat graph of W .

Theorem 6.1 (Postnikov [29]). *Let u and v be any two Weyl group elements. There exists a directed path from u to v in the quantum Bruhat graph. All shortest paths from u to v have the same degree $d_{\min}(u, v)$. The degree of any path from u to v is divisible by $d_{\min}(u, v)$.*

Moreover, there exists $w \in W$ such that

$$\text{GW}_{d_{\min}(u, \check{v}), 3}(\sigma_u, \sigma_v, \sigma_w) \neq 0,$$

and $d_{\min}(u, \check{v})$ is minimal with respect to this property, i.e., if there exists a degree d and $w \in W$ such that

$$\text{GW}_{d, 3}(\sigma_u, \sigma_v, \sigma_w) \neq 0$$

then

$$d_{\min}(u, \check{v}) \leq d.$$

We consider the following lemma whose proof we review for completeness.

Lemma 6.2. *For any positive root α , we always have that*

$$l(s_\alpha) \leq 2 \operatorname{ht}(\check{\alpha}) - 1.$$

Proof. First notice that for every simple root α

$$\frac{1}{2} \sum_{\gamma \in R^+} \langle \gamma, \check{\alpha} \rangle = 1$$

This is because

$$\sum_{\gamma \in R^+ - \{\alpha\}} \langle \gamma, \check{\alpha} \rangle = \sum_{\gamma \in R^+ - \{\alpha\}} \frac{\gamma - s_\alpha(\gamma)}{\alpha} = 0$$

and $\langle \alpha, \check{\alpha} \rangle = 2$. Thus, for any positive root α

$$\operatorname{ht}(\check{\alpha}) = \frac{1}{2} \sum_{\gamma \in R^+} \langle \gamma, \check{\alpha} \rangle$$

Now, for $w \in W$ let

$$I(w) := \{\beta \in R^+ : w(\beta) < 0\}$$

be the set of *inversions* of w . The second expression uses the order \leq defined on \mathfrak{t}^* where $a \leq b$ if and only if $b - a$ is a nonnegative linear combination of simple roots. Recall that $l(w) = |I(w)|$. Since s_α stabilizes the set $R^+ - I(s_\alpha)$

$$\sum_{\gamma \in R^+ - I(s_\alpha)} \langle \gamma, \check{\alpha} \rangle = \sum_{\gamma \in R^+ - I(s_\alpha)} \frac{\gamma - s_\alpha(\gamma)}{\alpha} = 0$$

For any root $\gamma \in I(s_\alpha)$ we have $s_\alpha(\gamma) = \gamma - \langle \gamma, \check{\alpha} \rangle \alpha < 0$, hence $\langle \gamma, \check{\alpha} \rangle \geq 1$ and

$$2 \operatorname{ht}(\check{\alpha}) = \sum_{\gamma \in R^+} \langle \gamma, \check{\alpha} \rangle = \sum_{\gamma \in I(s_\alpha)} \langle \gamma, \check{\alpha} \rangle = 2 + \sum_{\gamma \in I(s_\alpha) - \{\alpha\}} \langle \gamma, \check{\alpha} \rangle \geq l(s_\alpha) + 1.$$

□

Let $l_T : W \rightarrow \mathbb{Z}_{\geq 0}$ denote the word length function defined on W with respect to the generating set of reflections $\{s_\alpha\}_{\alpha \in R^+}$, i.e., for $w \in W$ if we write $w = s_{\alpha_1} \cdots s_{\alpha_r}$ for some positive roots $\alpha_1, \dots, \alpha_r$ and r is minimal, then $r = l_T(w)$. For $w \in W$, we call $l_T(w)$ the *absolute length* of w .

The following statement is a consequence of the previous Lemma and Theorem 6.1. It can be used to determine when a decomposition of w_0 into a product of reflections gives rise to a shortest path joining e and w_0 . We provide in the statement the corresponding upper bound for the Hofer-Zehnder capacity.

Theorem 6.3. *Let w_0 be the longest element in the Weyl group W with respect to the system of simple roots S . If there exist positive roots $\alpha_1, \dots, \alpha_r$ such that $w_0 = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_r}$ with $r = l_T(w_0)$ and*

$$\sum_{i=1}^r (2 \operatorname{ht}(\check{\alpha}_i) - 1) = l(w_0) = |R^+|,$$

then

$$d_{\min}(w_0, e) = \sum_{i=1}^r \check{\alpha}_i$$

In addition, if λ is in the interior of the Weyl chamber relative to the set of simple roots S ,

$$\min_d \omega_\lambda(d) = \sum_{i=1}^r \langle \lambda, \check{\alpha}_i \rangle$$

where the minimum is taken over all degrees of paths joining e with w_0 in the standard Bruhat graph of W . In particular we obtain the following upper bound for the Hofer-Zehnder capacity of $(\mathcal{O}_\lambda, \omega_\lambda)$

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \sum_{i=1}^r \langle \lambda, \check{\alpha}_i \rangle$$

Proof. For every $1 \leq i < r$, we have that

$$\begin{aligned} l(w_0) &\leq l(s_{\alpha_1} \cdot \dots \cdot s_{\alpha_i}) + l(s_{\alpha_{i+1}} \cdot \dots \cdot s_{\alpha_r}) \leq \sum_{j=1}^i l(s_{\alpha_j}) + \sum_{j=i+1}^r l(s_{\alpha_j}) \\ &\leq \sum_{j=1}^i (2 \operatorname{ht}(\check{\alpha}_j) - 1) + \sum_{j=i+1}^r (2 \operatorname{ht}(\check{\alpha}_j) - 1) = l(w_0) \end{aligned}$$

Thus for every $1 \leq i < r$,

$$l(s_{\alpha_1} \cdot \dots \cdot s_{\alpha_j}) = \sum_{i=1}^j (2 \operatorname{ht}(\check{\alpha}_i) - 1)$$

In particular

$$l(s_{\alpha_1} \cdot \dots \cdot s_{\alpha_i}) = l(s_{\alpha_1} \cdot \dots \cdot s_{\alpha_i} \cdot s_{\alpha_{i+1}}) + 1 - 2 \operatorname{ht}(\check{\alpha}_{i+1})$$

and

$$w_0 = s_1 \cdot s_2 \cdot \dots \cdot s_r \rightarrow \dots \rightarrow s_1 \cdot s_2 \rightarrow s_1 \rightarrow e$$

represents a directed path in the quantum Bruhat graph from w_0 to e . This path is a shortest path because of the minimality property of $r = l_T(w_0)$, and hence its degree equals $d_{\min}(w_0, e)$. \square

Remark 6.4. In section 8, we verify that for the longest element w_0 in the Weyl group W there exist positive roots $\alpha_1, \dots, \alpha_r$ such that $w_0 = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_r}$ with $r = l_T(w_0)$. Hence the assumptions made in the last theorem hold for any compact Lie group.

7. LOWER BOUNDS FOR THE HOFER-ZEHNDER CAPACITY AND HAMILTONIAN TORUS ACTIONS

In this section we describe how to compute lower bounds for the Hofer-Zehnder capacity of a symplectic manifold with a Hamiltonian torus action by using its moment map. By simplicity, we always assume that the points fixed by the torus action are isolated. We also estimate from below the Hofer-Zehnder capacity of coadjoint orbits of compact Lie groups as it was already done in the proof of Theorem 5.1 for coadjoint orbits of the unitary group.

Let T denote a torus. In what follows we always identify the Lie algebra $\mathfrak{u}(1)$ with \mathbb{R} . A *weight* of T is a Lie group morphism $\eta : T \rightarrow U(1)$. A *coweight* of T is a Lie group morphism $\xi : U(1) \rightarrow T$. Let $\Lambda \subset \mathfrak{t}$ be the kernel of the exponential map $\exp : \mathfrak{t} \rightarrow T$ and $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Z})$ be its dual. The differential of any weight $\mu : T \rightarrow U(1)$ is a Lie algebra morphism $\mathfrak{t} \rightarrow \mathfrak{u}(1) \cong \mathbb{R}$ that takes Λ into $2\pi\mathbb{Z}$. Conversely, any group morphism $\Lambda \rightarrow 2\pi\mathbb{Z}$ arises in this way. Thus, the set of weights $X^*(T) := \text{Hom}(T, U(1))$ can be identified with $2\pi\Lambda^* \subset \mathfrak{t}^*$. Similarly, the set of coweights $X_*(T) := \text{Hom}(U(1), T)$ is identified with $\frac{1}{2\pi}\Lambda \subset \mathfrak{t}$. In this section, we always see weights and coweights as elements of \mathfrak{t}^* and \mathfrak{t} , respectively. When we pair a coweight ξ with a weight η , we denote the composition $\eta \circ \xi$ by $\langle \eta, \xi \rangle$. We always identify $\text{Hom}(U(1), U(1))$ with \mathbb{Z} .

The Schur Lemma implies that for any representation V of T , we can write V as a direct sum

$$V = \bigoplus_{\eta \in X^*(T)} V_\eta$$

where $V_\eta = \{v \in V : t \cdot v = \eta(t)v \text{ for all } t \in T\}$. We call a η such that $V_\eta \neq \{0\}$ a weight of the representation. Any coweight ν of T defines a representation of $U(1)$ on V by pulling back the action of T on V and the weights of this representation are the nonzero elements of the set of integers $\{\langle \eta, \nu \rangle : \eta \in X^*(T)\}$.

Let (M, ω) be a symplectic manifold with a Hamiltonian action of a torus T generated by a moment map $\phi : M \rightarrow \mathfrak{t}^*$. The critical points of ϕ are the fixed points of the torus action. In this section, we assume that the number of fixed points is finite.

For every fixed point $p \in M$, the *isotropy weights at p* are the weights η_1, \dots, η_n such that the tangent space $T_p M$ is linearly symplectomorphic to the action on $(\mathbb{C}^n, \omega_{\text{st}})$ defined by

$$t \cdot (z_1, \dots, z_n) := (\eta_1(t)z_1, \dots, \eta_n(t)z_n)$$

and generated by the moment map

$$\begin{aligned} \mathbb{C}^n &\rightarrow \mathfrak{t}^* \\ (z_1, \dots, z_n) &\mapsto \frac{1}{2}(|z_1|^2 \eta_1 + \dots + |z_n|^2 \eta_n). \end{aligned}$$

An *isotropy weight of the torus action* is an isotropy weight of the torus action at some fixed point.

For $\xi \in \mathfrak{t}$, define

$$\begin{aligned}\phi^\xi : M &\rightarrow \mathbb{R} \\ p &\mapsto \langle \phi(p), \xi \rangle\end{aligned}$$

We call $\xi \in \mathfrak{t}$ *generic* if $\langle \eta, \xi \rangle \neq 0$ for every isotropy weight η of the torus action. In this case, the function $\phi^\xi : M \rightarrow \mathbb{R}$ is Morse and its set of critical points coincides with the set of points fixed by the torus action.

Let ξ be a coweight of T . The coweight ξ defines a Hamiltonian circle action on M with moment map $\phi^\xi : M \rightarrow \mathbb{R}$. If the isotropy weights of the torus action at a fixed point are η_1, \dots, η_n , the isotropy weights of the circle action defined by the coweight ξ are the integers $\langle \eta_1, \xi \rangle, \dots, \langle \eta_n, \xi \rangle$. The coweight ξ is generic if for every fixed point all the integers $\langle \eta_1, \xi \rangle, \dots, \langle \eta_n, \xi \rangle$ are nonzero.

Theorem 7.1. *Let (M^{2n}, ω) be a compact symplectic manifold with a Hamiltonian circle action S^1 generated by a moment map $H : M \rightarrow \mathbb{R}$. Assume that the number of fixed points is finite. Let $I \subset \mathbb{Z}$ be the set of all isotropy weights of the circle action and*

$$m^+ = \max_{m \in I} |m|.$$

Then the function

$$H' = \frac{1}{m^+} H : M \rightarrow \mathbb{R}$$

is slow and in particular

$$\text{osc } H' \leq c_{\text{HZ}}(M, \omega)$$

Proof. We show that the stabilizer subgroup of S^1 at every non fixed point is a cyclic subgroup of order less or equal to m^+ . This claim implies the theorem.

Let p be a fixed point of S^1 and m_1, \dots, m_n be the isotropy weights at p . The equivariant Darboux theorem asserts that there is a neighbourhood U of p in M equivariantly symplectomorphic to a neighborhood V of the origin in $(\mathbb{C}^n, \omega_{\text{st}})$ with the circle action defined by

$$t \cdot (z_1, \dots, z_n) = (t^{m_1} z_1, \dots, t^{m_n} z_n)$$

The stabilizer subgroup of $(z_1, \dots, z_n) \in V \setminus \{0\}$ is a cyclic group of order equal to $\gcd\{m_i : z_i \neq 0\}$. Note that $\gcd\{m_i : z_i \neq 0\}$ is less or equal to m^+ , and the claim holds at any point of the Darboux chart.

Finally, the stabilizer group of a point located anywhere in the manifold coincides with the stabilizer group of some point located at some equivariant Darboux's chart of some fixed point (see e.g. Guillemin, Lerman and Sternberg [13][Lemma 3.3.2]). The statement follows from the analysis done in the previous paragraphs. \square

Corollary 7.1. *Let (M^{2n}, ω) be a compact symplectic manifold with a Hamiltonian torus action T generated by a moment map $H : M \rightarrow \mathfrak{t}^*$. Let us assume that the set of fixed points of T are isolated. Let $I \subset X^*(T)$ be the set of isotropy weights of the torus action. For a coweight $\xi \in X_*(T)$, define*

$$m_\xi^+ := \max_{\eta \in I} |\langle \eta, \xi \rangle|$$

Then

$$\sup \left\{ \frac{1}{m_\xi^+} \text{osc}(\phi^\xi) : \text{generic } \xi \in X_*(T) \right\} \leq c_{\text{HZ}}(M, \omega)$$

We want to apply the previous Corollary to bound from below the Hofer-Zehnder capacity of coadjoint orbits of compact Lie groups. We use the same convention as in Section 3. Let G be a compact *simple* Lie group, $T \subset G$ be a maximal torus and W be the corresponding Weyl group. Let R be the corresponding system of roots relative to T and S be a choice of simple roots. For any positive root α , we write

$$\alpha = \sum_{\beta \in S} n_{\alpha\beta} \beta,$$

for some nonnegative integers $n_{\alpha\beta}$. We identify \mathfrak{g} and \mathfrak{g}^* via an adjoint invariant inner product (\cdot, \cdot) . Let $\lambda \in \mathfrak{t}_+^*$ be an element of the Weyl chamber relative to S and \mathcal{O}_λ be the coadjoint orbit passing through λ . The maximal group T acts hamiltonially on \mathcal{O}_λ with moment map $\phi : \mathcal{O}_\lambda \hookrightarrow \mathfrak{t}^*$ equals to the composition of the projection map $\mathfrak{g}^* \rightarrow \mathfrak{t}^*$ with the inclusion map $\mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$. The image of ϕ is the convex hull of $\{w(\lambda)\}_{w \in W}$. The set of all isotropy weights of the torus action of T on \mathcal{O}_λ is a subset of the set of roots and equal to the whole set of roots when \mathcal{O}_λ is a regular coadjoint orbit.

We say that $\alpha \leq \beta$ for two positive roots α and β , if $\beta - \alpha$ is a nonnegative linear combination of simple roots. The *highest positive root* is the positive root that is maximal with respect to the order that we define for positive roots. The existence and uniqueness of the highest positive root follows from the fact that G is simple.

In the next statement we give our lower bound for the Hofer-Zehnder capacity of coadjoint orbits. In the proof, we keep the notation of Corollary 7.1.

Theorem 7.2. *Let ρ be the highest positive root. Assume that the longest element w_0 in W relative to S can be decomposed as*

$$w_0 = s_{\alpha_1} \cdots s_{\alpha_r}$$

where $\alpha_1, \dots, \alpha_r$ are pairwise orthogonal positive roots. Then,

$$\max_{\alpha \in S} \left\{ \sum_{k=1}^r \frac{n_{\alpha_k} \alpha}{n_{\rho} \alpha} \langle \lambda, \check{\alpha}_k \rangle \right\} \leq c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda)$$

Proof. For $w \in W$, the weight decomposition of the tangent space of \mathcal{O}_λ at $w(\lambda)$ is

$$T_{w(\lambda)}\mathcal{O}_\lambda = \bigoplus_{\alpha \in R^+ - R_P^+} \mathfrak{g}_{-w(\alpha)}$$

The isotropy weights of the circle action defined by a coweight ξ is the set of integers

$$\{-\langle \alpha, \xi \rangle : \alpha \in W(R^+ - R_P^+)\}.$$

We call a coweight ξ positive if

$$\langle \alpha, \xi \rangle > 0$$

for every positive root α . We denote the set of positive coweights by $X_*(T)_+$. Every positive coweight ξ is generic by definition. We can always assume that a coweight is positive by taking a different system of simple roots if needed.

For a positive coweight ξ , the Morse index of $\phi^\xi : \mathcal{O}_\lambda \rightarrow \mathbb{R}$ at λ is the maximum possible and λ is a local maximum. Indeed, λ is an absolute maximum. Similarly, the absolute minimum of $\phi^\xi : \mathcal{O}_\lambda \rightarrow \mathbb{R}$ is achieved at $w_0(\lambda)$. Note that the value

$$m_\xi^+ = \max_{\alpha \in W(R - R_P)} |\langle \alpha, \xi \rangle|$$

is achieved when α is the highest positive root ρ .

Our orthogonality assumption implies that

$$w_0(\lambda) = \prod_{k=1}^r s_{\alpha_k}(\lambda) = \lambda - \sum_{k=1}^r \langle \lambda, \check{\alpha}_k \rangle \alpha_k$$

and for a positive coweight ξ

$$\text{osc}(\phi^\xi) = \sum_{k=1}^r \langle \lambda, \check{\alpha}_k \rangle \langle \alpha_k, \xi \rangle$$

Following Corollary 7.1, we want to maximize the expression

$$\frac{1}{\langle \rho, \xi \rangle} \text{osc}(\phi^\xi) = \sum_{k=1}^r \frac{\langle \alpha_k, \xi \rangle}{\langle \rho, \xi \rangle} \langle \lambda, \check{\alpha}_k \rangle$$

for $\xi \in X_*(T)_+$. The right hand side of the last equation is scale invariant and continuous as a function of the variable ξ on the convex cone

$$X_*(T)_+ \otimes \mathbb{R} = \{\xi \in \mathfrak{t} \setminus \{0\} : \langle \alpha, \xi \rangle \geq 0 \text{ for all } \alpha \in R^+\}$$

Thus,

$$\sup_{\xi \in X_*(T)_+} \left\{ \sum_{k=1}^r \frac{\langle \alpha_k, \xi \rangle}{\langle \rho, \xi \rangle} \langle \lambda, \check{\alpha}_k \rangle \right\} = \sup_{\xi \in X_*(T)_+ \otimes \mathbb{R}} \left\{ \sum_{k=1}^r \frac{\langle \alpha_k, \xi \rangle}{\langle \rho, \xi \rangle} \langle \lambda, \check{\alpha}_k \rangle \right\}$$

The change of variable

$$y := \frac{\xi}{\langle \rho, \xi \rangle}$$

transform our problem into the following linear optimization problem

$$\begin{cases} \text{Maximize} & \sum_{k=1}^r \langle \alpha_k, y \rangle \langle \lambda, \check{\alpha}_k \rangle \\ \text{Subject to} & \langle \rho, y \rangle = 1 \\ & \langle \alpha, y \rangle \geq 0 \text{ for all } \alpha \in S \end{cases}$$

The hyperplane in \mathfrak{t} defined by the equation $\langle \rho, y \rangle = 1$ cuts $X_*(T)_+ \otimes \mathbb{R}$ into the polytope

$$\Delta = \{y \in \mathfrak{t} : \langle \rho, y \rangle = 1, \langle \alpha, y \rangle \geq 0 \text{ for all } \alpha \in S\}.$$

The maximum value of the linear expression $\sum_{k=1}^r \langle \alpha_k, y \rangle \langle \lambda, \check{\alpha}_k \rangle$ on Δ is obtained at some of the vertices of the polytope Δ . Equivalently, the maximum value of the expression

$$\sum_{k=1}^r \frac{\langle \alpha_k, \xi \rangle}{\langle \rho, \xi \rangle} \langle \lambda, \check{\alpha}_k \rangle$$

is obtained at some of the one-dimensional faces of $X_*(T)_+ \otimes \mathbb{R}$. Each one-dimensional face of $X_*(T)_+ \otimes \mathbb{R}$ is spanned by some element in the basis dual to the basis of simple roots defined by the relation

$$(\tau_\alpha, \beta) = \delta_{\alpha, \beta} \text{ for any } \alpha, \beta \in S.$$

We conclude that

$$\sup_{\xi \in X_*(T)_+ \otimes \mathbb{R}} \left\{ \sum_{k=1}^r \frac{\langle \alpha_k, \xi \rangle}{\langle \rho, \xi \rangle} \langle \lambda, \check{\alpha}_k \rangle \right\} = \max_{\alpha \in S} \left\{ \sum_{k=1}^r \frac{\langle \alpha_k, \tau_\alpha \rangle}{\langle \rho, \tau_\alpha \rangle} \langle \lambda, \check{\alpha}_k \rangle \right\}$$

By Corollary 7.1, we get that

$$\max_{\alpha \in S} \left\{ \sum_{k=1}^r \frac{\langle \alpha_k, \tau_\alpha \rangle}{\langle \rho, \tau_\alpha \rangle} \langle \lambda, \check{\alpha}_k \rangle \right\} = \max_{\alpha \in S} \left\{ \sum_{k=1}^r \frac{n_{\alpha_k} \alpha}{n_\rho \alpha} \langle \lambda, \check{\alpha}_k \rangle \right\} \leq c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda)$$

□

Remark 7.3. The previous statement is compatible with Theorem 4.2, i.e., with the same notation as in the previous theorem, we have that

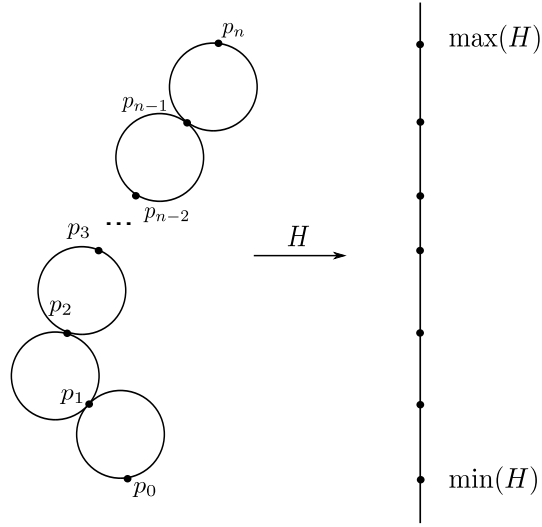
$$\max_{\alpha \in S} \left\{ \sum_{k=1}^r \frac{n_{\alpha_k} \alpha}{n_\rho \alpha} \langle \lambda, \check{\alpha}_k \rangle \right\} \leq \sum_{k=1}^r \langle \lambda, \check{\alpha}_k \rangle$$

More generally, let (M, ω) be a symplectic manifold and assume that S^1 acts Hamiltonially on (M, ω) with a moment map $H : M \rightarrow \mathbb{R}$. Assume that the fixed points of the circle action are isolated.

Let p_0, p_1, \dots, p_n be a sequence of fixed points such that every pair p_i, p_{i+1} of critical points in the sequence is joined by a S^1 -invariant sphere S_i and p_0, p_n are the minimum and maximum of H , respectively.

If the isotropy weight of the circle action restricted to the sphere S_i at p_i is k_i , then

$$|H(p_{i+1}) - H(p_i)| = k_i \omega(S_i)$$



(see e.g. McDuff and Tolman [24][Lemma 3.9]). Thus

$$\begin{aligned} \text{osc } H &= H(p_n) - H(p_0) \leq \sum_i |H(p_{i+1}) - H(p_i)| = \sum_i k_i \omega(S_i) \\ &\leq \max_i k_i \sum_i \omega(S_i) \leq m^+ \sum_i \omega(S_i), \end{aligned}$$

and

$$\frac{1}{m^+} \text{osc } H \leq \sum_i \omega(S_i),$$

where m^+ is defined as in Theorem 7.1. When the symplectic manifold is a coadjoint orbit and the circle action comes from a coweight of a maximal torus, the Hofer-Zehnder capacity of the coadjoint orbit is between the two values of the last inequality.

8. COMPUTATION OF BOUNDS HOFER-ZEHNDER CAPACITY

In this section we show that the assumptions made in Theorem 6.3 and Theorem 7.2 hold for any Weyl group and we compute for any compact simple Lie groups the corresponding bounds for the Hofer-Zehnder capacity of their regular coadjoint orbits.

We use the same convention as in Section 3. Let G be a compact *simple* Lie group and $T \subset G$ be a maximal torus. Let R be the set of roots associated with T and S be a choice of simple roots. We denote by W the corresponding Weyl group.

In the following theorem, we summarize the main results of the paper.

Theorem 8.1. *Let w_0 be the longest element of W relative to the set of simple roots S . There exist pairwise orthogonal positive roots $\alpha_1, \dots, \alpha_r$ such that*

$$l_T(w_0) = r,$$

$$w_0 = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_r}$$

and

$$\sum_{i=1}^r (2 \text{ht}(\check{\alpha}_i) - 1) = l(w_0) = |R^+|$$

In particular, for regular $\lambda \in \mathfrak{t}_+^*$ we obtain the following bounds for the Hofer-Zehnder capacity of the coadjoint orbit \mathcal{O}_λ with respect to its Kostant-Kirillov-Souriau form ω_λ

$$\max_{\alpha \in S} \left\{ \sum_{k=1}^r \frac{n_{\alpha_k} \alpha}{n_{\rho} \alpha} \langle \lambda, \check{\alpha}_k \rangle \right\} \leq c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \sum_{k=1}^r \langle \lambda, \check{\alpha}_k \rangle,$$

where ρ denotes the highest positive root.

We split the proof of the previous statement in several cases according to the type of the Lie group G . We provide the described decomposition of w_0 and the corresponding lower and upper bound for the Hofer-Zehnder capacity of the regular coadjoint orbit \mathcal{O}_λ . We omit the detailed calculations, although we give enough information so they can be verified by the reader.

Let λ be in the interior of the Weyl chamber relative to S , \mathcal{O}_λ be the coadjoint orbit passing through λ and ω_λ be the Kostant-Kirillov-Souriau form defined on \mathcal{O}_λ .

Type B. The standard root system for the group $B_n = SO(2n+1)$ is identified with the set of vectors $R = \{\pm e_i, \pm(e_j \pm e_k) : j \neq k\}_{1 \leq i, j \leq n} \subset \mathbb{R}^n$ with a choice of simple roots given by $S = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}$. The Dynkin diagram of B_n is shown in Figure 5

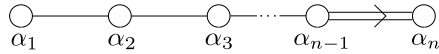


FIGURE 5. Dynkin diagram of B_n

The longest element w_0 of B_n seen as a map of \mathbb{R}^n is the reflection

$$\begin{aligned} \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ (x_1, \dots, x_n) &\mapsto (-x_1, \dots, -x_n) \end{aligned}$$

We have that $l_T(w_0) = n$, $l(w_0) = n^2$ and

$$w_0 = \begin{cases} s_{e_1-e_2} s_{e_1+e_2} s_{e_3-e_4} s_{e_3+e_4} \dots s_{e_{n-1}-e_n} s_{e_{n-1}+e_n} & \text{if } n \text{ is even} \\ s_{e_1-e_2} s_{e_1+e_2} s_{e_3-e_4} s_{e_3+e_4} \dots s_{e_{n-2}-e_{n-1}} s_{e_{n-2}+e_{n-1}} s_{e_n} & \text{if } n \text{ is odd} \end{cases}$$

Hence

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \begin{cases} 2\lambda_1 + 2\lambda_3 + \dots + 2\lambda_{n-1} & \text{if } n \text{ is even} \\ 2\lambda_1 + 2\lambda_3 + \dots + 2\lambda_{n-2} + 2\lambda_n & \text{if } n \text{ is odd} \end{cases}$$

The highest root ρ is $e_1 + e_2$ and

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq \max\{2\lambda_1, \lambda_1 + \lambda_2 + \cdots + \lambda_n\}$$

Type C. The standard root system for the group $C_n = Sp(n)$ is identified with the set of vectors $R = \{\pm 2e_i, \pm(e_j \pm e_k) : j \neq k\}_{1 \leq i, j \leq n} \subset \mathbb{R}^n$ with a choice of simple roots given by $S = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n\}$. The Dynkin diagram of C_n is shown in Figure 6

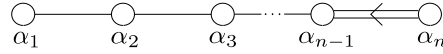


FIGURE 6. Dynkin diagram of C_n

Combinatorially speaking, the Weyl group C_n is the same as the Weyl group B_n , however the edges of its Bruhat graphs have different degrees.

As an automorphism of \mathbb{R}^n , the longest element w_0 of C_n is the reflection

$$\begin{aligned} \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ (x_1, \dots, x_n) &\mapsto (-x_1, \dots, -x_n) \end{aligned}$$

We have $l_T(w_0) = n, l(w_0) = n^2$ and

$$w_0 = s_{2e_1} s_{2e_2} \cdots s_{2e_n}$$

Hence,

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

The longest root is $\rho = 2e_1$ and

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Hence, we get the sharp expression for coadjoint orbits of type C

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Type D. The standard root system for the group $D_n = SO(2n)$ is identified with the set of vectors $R = \{\pm(e_j \pm e_k) : j \neq k\}_{1 \leq i, j \leq n} \subset \mathbb{R}^n$ with a choice of simple roots given by $S = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$. The Dynkin diagram of D_n is shown in Figure 7.

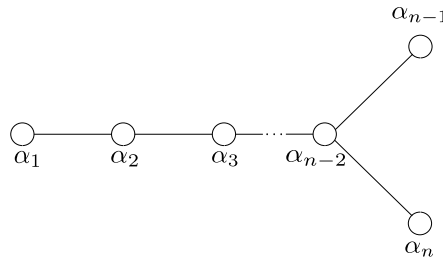


FIGURE 7. Dynkin diagram of D_n

As a map of \mathbb{R}^n , the longest element w_0 is the application

$$\mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$(x_1, \dots, x_{n-1}, x_n) \mapsto \begin{cases} (-x_1, \dots, -x_{n-1}, -x_n) & \text{if } n \text{ is even} \\ (-x_1, \dots, -x_{n-1}, x_n) & \text{if } n \text{ is odd} \end{cases}$$

We have that

$$l_T(w_0) = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases},$$

$l(w_0) = n(n-1)$, and

$$w_0 = \begin{cases} s_{e_1-e_2} s_{e_1+e_2} s_{e_3-e_4} s_{e_3+e_4} \cdots s_{e_{n-1}-e_n} s_{e_{n-1}+e_n} & \text{if } n \text{ is even} \\ s_{e_1-e_2} s_{e_1+e_2} s_{e_3-e_4} s_{e_3+e_4} \cdots s_{e_{n-2}-e_{n-1}} s_{e_{n-2}+e_{n-1}} & \text{if } n \text{ is odd} \end{cases}$$

Hence,

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \begin{cases} 2\lambda_1 + 2\lambda_3 + \dots + 2\lambda_{n-1} & \text{if } n \text{ is even} \\ 2\lambda_1 + 2\lambda_3 + \dots + 2\lambda_{n-2} & \text{if } n \text{ is odd} \end{cases}$$

On the other hand, $\rho = e_1 + e_2$ and

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq \begin{cases} \max\{2\lambda_1, \lambda_1 + \lambda_2 + \dots + \lambda_{n-1} + |\lambda_n|\} & \text{if } n \text{ is even} \\ \max\{2\lambda_1, \lambda_1 + \lambda_2 + \dots + \lambda_{n-1}\} & \text{if } n \text{ is odd} \end{cases}$$

Type E . There are three isomorphism classes of compact simple Lie groups of type E : E_6, E_7, E_8 . We start first with E_8 . A system of simple roots for E_8 as vectors in \mathbb{R}^8 is

$$S = \left\{ \alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \alpha_2 = e_1 + e_2, \alpha_3 = -e_1 + e_2, \right. \\ \left. \alpha_4 = -e_2 + e_3, \alpha_5 = -e_3 + e_4, \alpha_6 = -e_4 + e_5, \alpha_7 = -e_5 + e_6, \alpha_8 = -e_6 + e_7 \right\}$$

The Dynkin diagram of E_8 is shown in Figure 8.

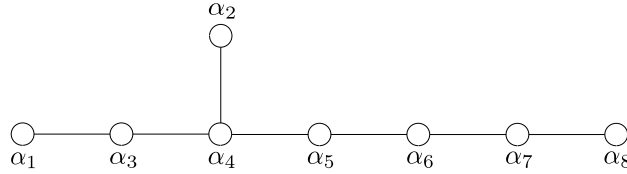


FIGURE 8. Dynkin diagram of E_8

As a map of \mathbb{R}^8 , the longest element of E_8 is the application

$$\mathbb{R}^8 \rightarrow \mathbb{R}^8$$

$$(x_1, \dots, x_8) \mapsto (-x_1, \dots, -x_8)$$

and its absolute length and length are equal to 8 and 120, respectively. We can write the longest element as the composition of reflections s_{r_1}, \dots, s_{r_7} and s_{r_8} where

$$\begin{aligned} r_1 &= -e_1 + e_2, r_2 = e_1 + e_2, r_3 = -e_3 + e_4, r_4 = e_3 + e_4, \\ r_5 &= -e_5 + e_6, r_6 = e_5 + e_6, r_7 = -e_7 + e_8, r_8 = e_7 + e_8 \end{aligned}$$

The upper bound for the Hofer-Zehnder capacity of a regular coadjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ of E_8 is given by

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq 2\lambda_2 + 2\lambda_4 + 2\lambda_6 + 2\lambda_8$$

The highest root equals to $e_7 + e_8$ and

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq \max\left\{2\lambda_8, \frac{1}{3}(\lambda_1 + \lambda_2 + \dots + \lambda_7 + 5\lambda_8)\right\}$$

We have finished our analysis for E_8 and now we continue with the one for E_7 . We keep the notation used in the previous paragraphs. A system of simple roots for E_7 is the set $\{\alpha_1, \alpha_2, \dots, \alpha_7\}$. Note that the Dynkin diagram of E_7 is contained in the Dynkin diagram of E_8 . The longest element of E_7 is the application

$$\begin{aligned} \mathbb{R}^8 &\rightarrow \mathbb{R}^8 \\ (x_1, \dots, x_6, x_7, x_8) &\mapsto (-x_1, \dots, -x_6, x_8, x_7), \end{aligned}$$

and its absolute length are equal to 7 and 63, respectively. We can write the longest element as the composition of the reflections $s_{r_1}, s_{r_2}, \dots, s_{r_7}$. Hence, the upper bound for the Hofer-Zehnder capacity of a regular coadjoint orbit of E_7 is

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq 2\lambda_2 + 2\lambda_4 + 2\lambda_6 + \lambda_8 - \lambda_7 = 2\lambda_2 + 2\lambda_4 + 2\lambda_6 - 2\lambda_7$$

The highest root is $-e_7 + e_8$ and

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq \max\left\{2\lambda_6 - 2\lambda_7, \frac{1}{2}(\lambda_1 + \dots + \lambda_6 - 4\lambda_7)\right\}$$

Finally, for E_6 a system of simple roots is $\{\alpha_1, \dots, \alpha_6\}$. The longest element of E_6 has absolute length and length equal to 4 and 36, respectively, and it can be written as the composition of the reflections $s_{t_1}, s_{t_2}, s_{t_3}$ and s_{t_4} where

$$\begin{aligned} t_1 &= -e_2 + e_3, t_2 = -e_1 + e_4, t_3 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8) \\ t_4 &= \frac{1}{2}(-e_1 - e_2 - e_3 - e_4 + e_5 - e_6 - e_7 + e_8) \end{aligned}$$

The upper bound for the Hofer-Zehnder capacity is given by

$$\begin{aligned} c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) &\leq -\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7 + \lambda_8 \\ &= -\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - 3\lambda_6 \end{aligned}$$

The highest root is $\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8)$ and the lower bound for the Hofer-Zehnder capacity is

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq \lambda_5 - \lambda_6 - \lambda_7 + \lambda_8 = \lambda_5 - 3\lambda_6$$

Type F . A system of simple roots for F_4 is

$$S = \{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}.$$

The Dynkin diagram of F_4 is shown in Figure 9.

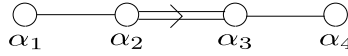


FIGURE 9. Dynkin diagram of F_4

The longest reflection w_0 in F_4 as a reflection of \mathbb{R}^4 is

$$\begin{aligned} \mathbb{R}^4 &\rightarrow \mathbb{R}^4 \\ (x_1, x_2, x_3, x_4) &\mapsto (-x_1, -x_2, -x_3, -x_4) \end{aligned}$$

We have that $l_T(w_0) = 4, l(w_0) = 24$ and

$$w_0 = s_{t_1} s_{t_2} s_{t_3} s_{t_4}$$

where

$$t_1 = e_1 + e_2, t_2 = e_1 - e_2, t_3 = e_3 + e_4, t_4 = e_3 - e_4$$

The upper bound for the Hofer-Zehnder capacity of a regular coadjoint orbit of type F_4 is

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq 2\lambda_1 + 2\lambda_3$$

The longest root of F_4 is $\rho = e_1 + e_2$. The lower bound for the Hofer-Zehnder capacity is

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq 2\lambda_1$$

Type G . Finally, a system of simple roots for G_2 is

$$S = \{\alpha_1 = e_1 - 2e_2 + e_3, \alpha_2 = e_2 - e_3\} \subset \mathbb{R}^3$$

and Dynkin diagram shown in Figure 10

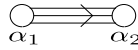


FIGURE 10. Dynkin diagram of G_2

We write

$$w_0 = s_{t_1} s_{t_2}$$

where

$$t_1 = e_2 - e_3, t_2 = 2e_1 - e_2 - e_3$$

Hence,

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \frac{2}{3}(\lambda_1 - \lambda_2 - 2\lambda_3) = \frac{2}{3}(3\lambda_1 + \lambda_2)$$

The highest root is $\rho = 2e_1 - e_2 - e_3$, and

$$c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \geq \frac{2}{3}(2\lambda_1 + \lambda_2)$$

All the bounds for the Hofer-Zehnder capacity of coadjoint orbits are summarized in the following table

| G | Lower bound | Upper bound |
|----------|---|--|
| $U(n)$ | $\frac{1}{2} \sum_{i=1}^n \lambda_i - \lambda_{n-i+1} $ | $\frac{1}{2} \sum_{i=1}^n \lambda_i - \lambda_{n-i+1} $ |
| $Sp(2n)$ | $\lambda_1 + \cdots + \lambda_n$ | $\lambda_1 + \cdots + \lambda_n$ |
| $SO(n)$ | | |
| $n = 4m$ | $\lambda_1 + \cdots + \lambda_n , 2\lambda_1$ | $2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_{n-1}$ |
| $4m + 1$ | $\lambda_1 + \cdots + \lambda_n, 2\lambda_1$ | $2\lambda_1 + 2\lambda_3 \cdots + 2\lambda_{n-1}$ |
| $4m + 2$ | $\lambda_1 + \cdots + \lambda_{n-1}, 2\lambda_1$ | $2\lambda_1 + 2\lambda_3 + \cdots + 2\lambda_{n-2}$ |
| $4m + 3$ | $\lambda_1 + \cdots + \lambda_n, 2\lambda_1$ | $2\lambda_1 + 2\lambda_3 \cdots + 2\lambda_n$ |
| E_6 | $\lambda_5 - \lambda_6 - \lambda_7 + \lambda_8$ | $\lambda_3 + \lambda_4 + \lambda_5 - \lambda_1 - \lambda_2 - 3\lambda_6$ |
| E_7 | $\frac{1}{2}(\lambda_1 + \cdots - 4\lambda_7), 2\lambda_6 - 2\lambda_7$ | $2\lambda_2 + 2\lambda_4 + 2\lambda_6 - 2\lambda_7$ |
| E_8 | $\frac{1}{3}(\lambda_1 + \cdots + 5\lambda_8), 2\lambda_8$ | $2\lambda_2 + 2\lambda_4 + 2\lambda_6 + 2\lambda_8$ |
| F_4 | $2\lambda_1$ | $2\lambda_1 + 2\lambda_3$ |
| G_2 | $\frac{2}{3}(2\lambda_1 + \lambda_2)$ | $\frac{2}{3}(3\lambda_1 + \lambda_2)$ |

Remark 8.2. Note that regardless of our bounds being sharp or not, we always get the following inequality

$$\frac{2}{3} \sum_{k=1}^r \langle \lambda, \check{\alpha}_k \rangle \leq c_{\text{HZ}}(\mathcal{O}_\lambda, \omega_\lambda) \leq \sum_{k=1}^r \langle \lambda, \check{\alpha}_k \rangle$$

9. ACKNOWLEDGMENTS

I would like to thank Yael Karshon and Leonid Polterovich for useful discussions. This research is supported by the Israel Science Foundation grants 178/13 and 1380/13.

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